Examination, Clarification, and Simplification of Modal Decoupling Method for Multiconductor Transmission Lines

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***Abstract*—In the application of the modal decoupling method, questions arise as to why the nonnormal matrices LC and CL are diagonalizable. Is the definition of the characteristic impedance matrix $\mathbf{Z}_c$ unique? Is it possible to normalize current and voltage eigenvectors simultaneously, yet assure the correct construction of the $\mathbf{Z}_c$ matrix? Under what conditions do $\mathbf{M}_i^t\mathbf{M}_v = \mathbf{I}$ and $\mathbf{Z}_c = \mathbf{M}_v\mathbf{M}_i^{-1}$? In this paper, these questions are thoroughly addressed. We will prove the diagonalizability of matrices LC and CL for lossless transmission lines (though the diagonalizability of their complex analogues, ZY and YZ matrices, is not guaranteed for lossy lines), and will demonstrate the properties of their eigenvalues. We have developed an algorithm to decouple one type of matrix differential equation, and to construct the characteristic impedance matrix $\mathbf{Z}_c$ explicitly and efficiently. Based on this work, the congruence and similarity transformations, which have caused considerable confusion and not a few errors in the decoupling and solution of the matrix telegrapher's equations, will be analyzed and summarized. In addition, we will also demonstrate that under certain conditions, the diagonalization of two or more matrices by means of the congruence or similarity transformations may lead to coordinate system "mismatch" and introduce erroneous results.**

I. Introduction

The even- and odd-mode decomposition method and the c- and π-mode decomposition method, can correctly solve electromagnetic coupling problems involving two symmetrical and two asymmetrical lines, but not more complex structures. The modal decoupling technique is a powerful extension of these two methods, in that it handles an arbitrary number of coupled lines at arbitrary locations. This technique has been applied to the analysis of multiconductor transmission line (MTL) problems for more than two decades [1]–[10]. In 1973, Marx [2] applied modal analysis to second order matrix differential equations and computed the characteristic impedance of the MTL's using voltage and current eigenvectors of the **LC** and **CL** matrices, where **L** is the inductance matrix and **C** is the capacitance matrix of the interconnect structure. In his work, Marx proved the bi-orthogonality between the voltage and current eigenvectors. Djordjevic *et al.* [4]–[10] have employed this method to solve various problems involving MTL's and networks in both the time and frequency domains.

With the increasing complexity of digital electronic systems and decreasing rise/fall times of the data pulses propagating through these systems, the behavior of MTL networks has become a new design topic for digital design engineers. As a result, the modal decoupling method has become one of the most popular approaches in the analysis of signal integrity, including waveform distortion, multiple reflections, and crosstalk. The advantages of the modal decoupling method include its simplicity of implementation and its ability to handle the complex geometries of real-world physical problems. Nonetheless, rigorous evaluations of the mathematical support of this technique have not been reported in the literature. The characteristic impedance matrix of a transmission line system is constituted from the voltage and current eigenvectors. However, the norms of these eigenvectors are not unique. Without other constraints, this type of construction will lead to nonunique definitions of the $\mathbf{M}_v$, $\mathbf{M}_i$, and $\mathbf{Z}_c$ matrices. Kajfez [11] first showed that the characteristic impedance matrix of an MTL system can be constructed from the voltage eigenvectors with prespecified norms (canonical norms). In Kajfez's approach, the telegrapher's equations are converted, in terms of the parameter matrices **C** and **L**, into one-sided (both matrices **L** and **C** are on one side of the equation) and two-sided (one matrix is on one side of the equation and the other matrix is on the opposite side of the equation) matrix differential equations, as will be presented in the next section. The modal decomposition technique was applied by Kajfez to solve the two-sided form of matrix equation (others have used the modal decomposition technique to solve the one-sided form of matrix equation, whether or not it was actually correct to do so). Kajfez cleverly borrowed techniques from quantum mechanics and linear algebra and applied them to electrical engineering applications.

However, the approach Kajfez used in [11] to decouple the matrix differential equations and to find the canonical norms is lengthy. In this article, we shall describe an algorithm for decoupling this type of equation which generates correct results directly and provides physical insight into the modal decoupling technique. As a starting point, here we only consider lossless transmission lines with real impedance and admittance matrices in the frequency domain. More general cases of lossy transmission lines with complex parameter matrices employing

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the generalized modal decoupling technique will be presented at a later time.

The remainder of this paper is organized as follows: In Section II, we shall derive the simplified method for decoupling the two-sided matrix differential equations and show how the algorithm can be used to compute the canonical norms of the $\mathbf{M}_v$ matrix. Section III investigates various conventional approaches for the solution of one-sided matrix differential equations, and illustrates potential problems with improperly specified norms. In Section IV, examples are provided which demonstrate the cases where erroneous results may occur, when incorrect diagonalization procedures are applied to matrices.

II. A New Algorithm for Decoupling Two-Sided Matrix Differential Equations

In this section, a simultaneous diagonalization of two symmetric matrices based on two successive transformations will be developed and then applied to decouple the two-sided matrix second-order differential equations.

The matrix telegrapher's equations, which govern the voltage and current distributions along the lossless MTL's, are

$$\frac{d}{dz}|V\rangle = -j\omega\mathbf{L}|I\rangle \tag{1}$$

$$\frac{d}{dz}|I\rangle = -j\omega\mathbf{K}|V\rangle \tag{2}$$

where z is a spatial coordinate, $\mathbf{L}$ is the inductance matrix and $\mathbf{K}$ (or $\mathbf{C}$ in some references) is the capacitance (or more precisely, the static induction) matrix representing the stored magnetic and electric energy in a passive network, respectively. The $n \times n$ $\mathbf{L}$ and $\mathbf{K}$ matrices are real, symmetric, and positive definite [11], where n represents the number of transmission lines. The unknowns, $|V\rangle$ and $|I\rangle$, are respectively the voltage and current vectors in the corresponding n-dimensional inner product space.[1] Taking the derivative with respective to z in (1), and eliminating $\frac{d}{dz}|I\rangle$ by means of (2), we obtain

$$\mathbf{L}^{-1}\frac{d^2}{dz^2}|V\rangle = -\omega^2\mathbf{K}|V\rangle. \tag{3}$$

Similarly, (2) can be written as

$$\mathbf{K}^{-1}\frac{d^2}{dz^2}|I\rangle = -\omega^2\mathbf{L}|I\rangle. \tag{4}$$

premultiplying (3) and (4) by L and K, respectively, we arrive at

$$\frac{d^2}{dz^2}|V\rangle = -\omega^2\mathbf{LK}|V\rangle \tag{5}$$

and

$$\frac{d^2}{dz^2}|I\rangle = -\omega^2\mathbf{KL}|I\rangle. \tag{6}$$

Since the matrices $\mathbf{L}^{-1}$ and $\mathbf{K}^{-1}$ are also real symmetric and positive definite, matrices $\mathbf{L}^{-1}$ and $\mathbf{K}$ in (3), as well as $\mathbf{K}^{-1}$ and $\mathbf{L}$ in (4), may be converted into diagonal forms simultaneously by congruence transformations [13], [14]. Let $\mathbf{A} = \mathbf{L}^{-1}$ and $\mathbf{B} = \mathbf{K}$. To diagonalize the matrices $\mathbf{A}$ and $\mathbf{B}$ simultaneously, consider the generalized eigen-equation corresponding to (3)

$$(\mathbf{B} - \lambda_i\mathbf{A})|x_i\rangle = 0 \tag{7}$$

where $|x_i\rangle$ is the i-th generalized eigenvector of (7) and λ_i is the i-th root of the equation

$$\det(\mathbf{B} - \lambda_i\mathbf{A}) = 0.$$

We will construct linearly independent eigenvectors $\{|e_i'\rangle\}$ which satisfy the generalized eigen-equation (7). Since $\mathbf{A}$ is Hermitian and positive definite, we may solve

$$\mathbf{A}|\varphi_j\rangle = \alpha_j|\varphi_j\rangle$$

for a complete orthonormal set of eigenvectors $\{|\varphi_j\rangle\}$ with real and positive eigenvalues $\{\alpha_j\}$. First, we construct a matrix $\mathbf{S}$ with $\alpha_j^{-\frac{1}{2}}|\varphi_j\rangle$ as its columns, namely

$$\mathbf{S} = [\alpha_j^{-\frac{1}{2}}|\varphi_j\rangle]$$

so that

$$\mathbf{S}^t\mathbf{S} = [\alpha_j^{-1}\delta_{ij}]$$

and

$$\det\mathbf{S}^t\mathbf{S} \equiv |\det\mathbf{S}|^2 = \prod_{j=1}^{n}\alpha_j^{-1} \neq 0.$$

Therefore, $\mathbf{S}$ is nonsingular, so that $\mathbf{S}^{-1}$ exists. Furthermore, it may be observed that

$$\begin{aligned}\mathbf{S}^t\mathbf{AS} &= [\alpha_i^{-\frac{1}{2}}\langle\varphi_i|]\mathbf{A}[\alpha_j^{-\frac{1}{2}}|\varphi_j\rangle] \\ &= [\alpha_i^{-\frac{1}{2}}\langle\varphi_j|\alpha_j^{-\frac{1}{2}}\alpha_j|\varphi_j\rangle] \\ &= [\langle\varphi_i|\varphi_j\rangle] = \mathbf{I}\end{aligned} \tag{8}$$

where $\mathbf{I}$ is the identity matrix.

We now define

$$\mathbf{M} \equiv \mathbf{S}^t\mathbf{BS} = \mathbf{M}^t. \tag{9}$$

Since $\mathbf{M}$ is Hermitian, we may solve

$$\mathbf{M}|\psi_i\rangle = \lambda_i|\psi_i\rangle \tag{10}$$

for a complete orthonormal set of eigenvectors $\{|\psi_i\rangle\}$ with real eigenvalues $\{\lambda_i\}$. Then we construct

$$|e_i'\rangle = \mathbf{S}|\psi_i\rangle \tag{11}$$

or

$$|\psi_i\rangle = \mathbf{S}^{-1}|e_i'\rangle. \tag{12}$$

Using (8), (9), and (12), we can rewrite (10) as

$$(\mathbf{S}^t\mathbf{BS} - \lambda_i\mathbf{S}^t\mathbf{AS})\mathbf{S}^{-1}|e_i'\rangle = 0 \tag{13}$$

which, premultiplying by $(\mathbf{S}^t)^{-1}$, yields

$$(\mathbf{B} - \lambda_i\mathbf{A})|e_i'\rangle = 0. \tag{14}$$

Thus, we have proven that $\{|e_i'\rangle\}$ are the eigenvectors of the generalized eigen-equation (7). Because $\mathbf{S}$ is nonsingular, (12) indicates that $|e_i'\rangle$ for $i = 1, 2, \ldots, n$, in (14) are linearly independent and form a basis. If they were not, there would

[1] In classical notation [12], a ket vector $|\,\rangle$ represents a column vector in the n-dimensional vector space X, while a bra vector $\langle\,|$ represents a row vector in the corresponding dual space X^*. The inner product is defined as the canonical product of vectors in the spaces X and X^*, i.e., $(|y\rangle, |x\rangle) = \langle y|x\rangle$.

exist a set of constants $\{\alpha_j\}$ not all zero such that

$$\sum_{j=1}^{n} \alpha_j |e'_j\rangle = 0$$

which by (12) would imply

$$\sum_{j=1}^{n} \alpha_j |\psi_j\rangle = 0$$

for $\{\alpha_j\}$ not all zero, a contradiction to the independency of $\{|\psi_j\rangle\}$.

Since $\mathbf{M}$ is Hermitian, there exists a unitary matrix $\mathbf{U}$ such that

$$\mathbf{U}^t\mathbf{M}\mathbf{U} = \mathbf{D}_M$$

where $\mathbf{D}_M$ is a diagonal matrix with the eigenvalues of $\mathbf{M}$ as its diagonal elements. Note that matrix $\mathbf{U}$ is orthonormal, while matrix $\mathbf{S}$ is orthogonal, but not normalized. Here and throughout the paper, an orthogonal matrix is different from the conventional definition by which its column vectors are normalized, and a normalized vector means a vector with magnitude of unity.

We now construct a matrix $\mathbf{Q} \equiv \mathbf{S}\mathbf{U}$. For simplicity, these two successive changes of the bases are merged into one, providing the following relations

$$\mathbf{Q}^t\mathbf{A}\mathbf{Q} = \mathbf{U}^t\mathbf{S}^t\mathbf{A}\mathbf{S}\mathbf{U} = \mathbf{U}\mathbf{U}^t = \mathbf{I} \tag{15}$$

and

$$\mathbf{Q}^t\mathbf{B}\mathbf{Q} = \mathbf{U}^t\mathbf{S}^t\mathbf{B}\mathbf{S}\mathbf{U} = \mathbf{U}^t\mathbf{M}\mathbf{U} = \mathbf{D}_M. \tag{16}$$

From (15) and (16), it may be observed that the first congruence transformation, $\mathbf{S}^t\mathbf{A}\mathbf{S}$, has transformed $\mathbf{A}$ into an identity matrix, while the consecutive transformation, $\mathbf{U}^t\mathbf{I}\mathbf{U}$, keeps the identity matrix unchanged. On the other hand, the first transformation, $\mathbf{S}^t\mathbf{B}\mathbf{S}$, preserves the symmetric properties of $\mathbf{B}$, while the consecutive unitary transformation converts $\mathbf{B}$ into a diagonal form. Note that when $\mathbf{A}$ and $\mathbf{B}$ are transformed into the basis $\{|e'_i\rangle\}$ by $\mathbf{Q}$, in general, the diagonal elements of matrices $\mathbf{I}$ and $\mathbf{D}_M$ are not the eigenvalues of the matrices $\mathbf{A}$ and $\mathbf{B}$, and the column vectors of $\mathbf{Q}$ are neither eigenvectors of $\mathbf{A}$ nor $\mathbf{B}$, but are eigenvectors of matrix $\mathbf{A}^{-1}\mathbf{B}$ or $\mathbf{LK}$. This transformation that diagonalizes the matrix $\mathbf{A}^{-1}\mathbf{B}$ is not a unitary one. Furthermore, the ij-th elements of $\mathbf{A}$ in the $|e'_i\rangle$ basis is

$$\langle e'_i|\mathbf{A}|e'_j\rangle = \langle\psi_i|\mathbf{S}^t\mathbf{A}\mathbf{S}|\psi_j\rangle = \delta_{ij}.$$

The above equation is the component form of (15). Note that $|e'_i\rangle$ and $|e'_j\rangle$ are not orthonormal with respect to the identity matrix, but are orthonormal with respect to the kernel $\mathbf{A}$ for $i = 1, 2, \ldots, n$, and the inner product of $|e'_i\rangle$ and $|e'_j\rangle$ is not equal to δ_{ij}, unless $\mathbf{A}$ becomes an identity matrix. Using (14), we have

$$\langle e'_i|\mathbf{B}|e'_j\rangle = \lambda_i\langle e'_i|\mathbf{A}|e'_j\rangle = \lambda_i\delta_{ij}.$$

This equation is the component form of (16).

The matrices $\mathbf{A}$ and $\mathbf{B}$ are not simultaneously similar to two diagonal matrices, but are simultaneously congruent to two diagonal matrices. Notice that the diagonalization of $\mathbf{A}$ and $\mathbf{B}$ by congruence transformations is independent of the degeneracy of the eigenvalues $\{\lambda_i\}$.

The aforementioned features, which are associated with the linear transformations in the Euclidean space, will be used in Section IV. We will now demonstrate explicitly how to construct matrix $\mathbf{Q}$ directly from $\mathbf{A}$ and $\mathbf{B}$ without passing through the two consecutive steps. We first search for $\{\lambda_i\}$ as the roots of the equation

$$\det(\mathbf{B} - \lambda_i\mathbf{A}) = 0.$$

We then solve (14) for eigenvectors $\{|e'_i\rangle\}$ corresponding to λ_i, and scale these eigenvectors such that

$$\langle e'_i|\mathbf{A}|e'_j\rangle = \delta_{ij}.$$

The vectors $\{|e'_i\rangle\}$ are simply the columns of matrix $\mathbf{Q}$. Under the conditions that $\mathbf{A}$ and $\mathbf{B}$ are real and symmetric for lossless lines, matrix $\mathbf{Q}$ may be chosen to be real. If a degeneracy occurs, the eigenvectors $\{|e'_i\rangle\}$ may be chosen such that $\mathbf{Q}^{-1}$ exists.

Now we are ready to apply the aforementioned transformations to matrix differential equations. To bring (3) into a proper basis system, we represent voltage vector $|V\rangle$ in the $\{|e'_i\rangle\}$ basis by

$$|V'\rangle = \mathbf{Q}^{-1}|V\rangle.$$

This coordinate transformation of $|V\rangle$ is applicable due to the existence of $\mathbf{Q}^{-1}$. As a matter of fact, the columns of $\mathbf{Q}$ are $\{|e'_i\rangle\}$, which are linearly independent. Expressing $|V\rangle$ in terms of $|V'\rangle$, and premultiplying (3) by $\mathbf{Q}^t$, we have

$$\frac{d^2}{dz^2}|V'\rangle = -\omega^2\mathbf{D}_M|V'\rangle \tag{17}$$

where the diagonal matrix $\mathbf{D}_M$ consists of eigenvalues $\lambda_i = \nu_i^{-2}$ as its elements. We now define the modal propagation constants β_i as

$$\beta_i^2 = \frac{\omega^2}{\nu_i^2}$$

where $i = 1, 2, \ldots, n$ and ν_i is the i-th modal velocity. Hence, each decoupled differential equation in (3) has the general solution

$$V'_i = a_i^+ e^{-j\beta_i z} + a_i^- e^{j\beta_i z} \tag{18}$$

where the amplitudes of the modal voltages a_i^+ and a_i^- of the forward and reflected waves at two given locations are determined by two-point boundary conditions [15].

Since (17) is represented in the modal coordinate system, the boundary conditions cannot be directly applied. After transforming $|V'\rangle$ back to the original basis by performing the inverse transformation shown in [11], we obtain

$$|V\rangle = \sum_{i=1}^{n}(a_i^+ e^{-j\beta_i z} + a_i^- e^{j\beta_i z})|e'_i\rangle \tag{19}$$

where $|e'_i\rangle$ is the i-th column of $\mathbf{Q}$. Similarly, the general solution of (6), $|I\rangle$ and its basis $\{|i'_i\rangle\}$, can be obtained by decoupling (6) using the aforementioned procedures. We substitute (19) into (1), and obtain

$$|I\rangle = \sum_{i=1}^{n}(a_i^+ e^{-j\beta_i z} - a_i^- e^{j\beta_i z})|i'_i\rangle \tag{20}$$

where

$$|i_i'\rangle = \frac{1}{\lambda i}\mathbf{L}^{-1}|e_i'\rangle. \tag{21}$$

Thus, the two sets of eigenvectors, $\{|e_i'\rangle\}$ and $\{|i_i'\rangle\}$, are uniquely determined and exhibit the bi-orthogonal property

$$\langle e_i'|i_j'\rangle = \langle e_i'|\mathbf{L}^{-1}|e_j'\rangle\frac{1}{\lambda_j} = \delta_{ij}\frac{1}{\lambda_j}.$$

Now we are in a position to define two bi-orthonormal vectors

$$|e_i^c\rangle = \lambda_i^{\frac{1}{2}}|e_i'\rangle$$

and

$$|i_i^c\rangle = \lambda_i^{\frac{1}{2}}|i_i'\rangle$$

where the norms of $|e_i^c\rangle$ and $|i_i^c\rangle$ are defined as their canonical norms. The current $|I\rangle$ given by (20) is different from the $|I\rangle$ solved directly from (6). In fact, solving the two coupled telegrapher's (1) and (2) is different from solving the two decoupled Helmholtz (5) and (6). Even though the two Helmholtz equations are derived from the telegrapher's equations, the Helmholtz equations have an enlarged solution domain in comparison to the solution domain of the telegrapher's (1) and (2) or (1) and (5). This difference appears as a result of the elimination of the constraint between the dependent variables $|V\rangle$ and $|I\rangle$. The use of (1) provides the necessary constraint between $|V\rangle$ and $|I\rangle$ and ensures the correct construction and the unique determination of the $\mathbf{Z}_c$ matrix.

Thus far, we have proven that $\{|e_i^c\rangle\}$ and $\{|i_i^c\rangle\}$ as well as $\{|e_i\rangle\}$ and $\{\langle i_i|\}$ form a complete set of nonorthogonal bases in the n-dimensional inner product space and its dual space, where $\{|e_i\rangle\}$ and $\{\langle i_i|\}$ are the arbitrarily-normed voltage and current eigenvector sets, respectively. Every vector $\{\langle i_i|\}$ is orthonormal to every vector $\{|e_j\rangle\}$, for $i \neq j$. This result is supported by a theorem that states that if $\{\langle i_i|\}$ is a basis in an n-dimensional vector space, then there is a unique basis $\{|e_j\rangle\}$ in its dual space with the property that $\langle i_i|e_j\rangle = \delta_{ij}$ [16].

Recall that the characteristic impedance matrix, $\mathbf{Z}_c$, is defined as $|V_f\rangle = \mathbf{Z}_c|I_f\rangle$, and $|V_f\rangle$ and $|I_f\rangle$ can be expressed as

$$|V_f\rangle = \sum_{i=1}^{n}(A_i^+e^{-j\beta_i z})|e_i^c\rangle \quad \text{and} \quad |I_f\rangle = \sum_{i=1}^{n}(A_i^+e^{-j\beta_i z})|i_i^c\rangle \tag{22}$$

where $A_i^+ = \frac{a_i^+}{\lambda_i^{\frac{1}{2}}}$.

Due to the bi-orthonormality of $\{|e_i^c\rangle\}$ and $\{|i_i^c\rangle\}$, and the constraint between $|V\rangle$ and $|I\rangle$, the "projection" of the forward voltage wave $|V_f\rangle$ onto $|i_i\rangle$ is the same as the "projection" of the forward current wave $|I_f\rangle$ onto $|e_i^c\rangle$, namely

$$\langle e_i^c|I_f\rangle = \langle i_i^c|V_f\rangle = A_i^+e^{-j\beta_i z}. \tag{23}$$

Substituting (23) into (22) respectively, we obtain

$$|V_f\rangle = \sum_{i=1}^{n}\langle e_i^c|I_f\rangle|e_i^c\rangle = \sum_{i=1}^{n}|e_i^c\rangle\langle e_i^c|I_f\rangle = \mathbf{Z}_c|I_f\rangle$$

and

$$|I_f\rangle = \sum_{i=1}^{n}\langle e_i^c|I_f\rangle|i_i^c\rangle = \sum_{i=1}^{n}|i_i^c\rangle\langle e_i^c|I_f\rangle.$$

Thus, the two important equations can be readily established as [11]

$$\mathbf{Z}_c = \sum_{i=1}^{n}|e_i^c\rangle\langle e_i^c| \quad \text{and} \quad \sum_{i=1}^{n}|i_i^c\rangle\langle e_i^c| = \mathbf{I}$$

where $\mathbf{Z}_c$ and the identity matrix, $\mathbf{I}$, are expressed as the finite sums of the outer products of the bases $\{|e_i^c\rangle\}$ and the $\{|i_i^c\rangle\}$. Constructing $\mathbf{M}_v$ using $|e_i^c\rangle$ as its i-th column and $\mathbf{M}_i$ using $|i_i^c\rangle$ as its i-th column, then

$$\mathbf{Z}_c = \mathbf{M}_v\mathbf{M}_v^t = \mathbf{M}_v\mathbf{M}_i^{-1} \tag{24}$$

and

$$\mathbf{M}_i\mathbf{M}_v^t = \mathbf{I}. \tag{25}$$

Equation (24) indicates that the characteristic impedance, $\mathbf{Z}_c$, is uniquely defined, and (25) exhibits the bi-orthonormality between $\mathbf{M}_v$ and $\mathbf{M}_i$.

So far, we have completed detailed derivations that support the simultaneous diagonalization of matrices $\mathbf{L}^{-1}$ and $\mathbf{K}$, and the decoupling of the matrix differential equations in the n-dimensional inner product space. The detailed algorithm for solving the two-sided matrix equations has been outlined, and the useful components of this result have been described in this section. We would like to underscore the following points:

- To diagonalize matrices $\mathbf{L}^{-1}$ and $\mathbf{K}$ simultaneously, the relationship $\langle e_i'|\mathbf{L}^{-1}|e_j'\rangle = \delta_{ij}$ is enforced first.
- To assure that both $|V\rangle$ and $|I\rangle$ are the general solutions to (1) and (2) and that the bi-orthonormality between the $\{|e_i^c\rangle\}$ and the $\{|i_i^c\rangle\}$ is satisfied, the component form of (15) needs to be modified as $\langle e_i^c|\mathbf{L}^{-1}|e_j^c\rangle = \delta_{ij}\lambda_i$, where λ_i are the roots of

$$\det(\mathbf{B} - \lambda\mathbf{A}) = 0.$$

Only in this way can the bi-orthonormality of the voltage and the current eigenvectors, and the constraint between $|V\rangle$ and $|I\rangle$, be satisfied.

III. Analysis of Techniques for the Decoupling of One-Sided Matrix Differential Equations

As we mentioned in the Introduction, Marx applied the modal decoupling method to the one-sided matrix differential equation and proved the bi-orthogonality between the voltage and current eigenvectors [2]. Following his ideas as well as his notation, many authors have employed the modal decoupling method, of which some implementations have been correct, while some have not. Because errors have arisen, the underpinning theory is worth clarifying rigorously.

In Marx's approach, (5) was intended to be decoupled by applying a similarity transformation to matrix $\mathbf{LK}$. In the Euclidean space, due to the symmetry and the positive definiteness of matrices $\mathbf{L}$ and $\mathbf{K}$, the matrix $\mathbf{LK}$ is similar to a diagonal matrix. However, in the unitary space, $\mathbf{ZY}$, the complex analogue of $\mathbf{LK}$ in the presence of losses, is similar, in general, to a Jordan canonical matrix [3], [17].

In this section we shall demonstrate the diagonalizability of the asymmetric matrix $\mathbf{LK}$, thereby confirming the correctness of Marx's approach. premultiplying (14) by $\mathbf{A}^{-1}$, we obtain

$$(\mathbf{A}^{-1}\mathbf{B} - \lambda_i \mathbf{I})|e_i'\rangle = 0. \tag{26}$$

This is the standard eigen-equation of $\mathbf{A}^{-1}\mathbf{B}$, with $|e_i'\rangle$ being the i-th eigenvector of matrix $\mathbf{A}^{-1}\mathbf{B}$ or $\mathbf{LK}$. Moreover, it was proven in (14) of the previous section that eigenvectors $\{|e_i'\rangle\}$ form a basis, namely, that all eigenvectors in the set $\{|e_i'\rangle\}$ are linearly independent. Since (14) and (26) are equivalent, we have thus proven that matrices $\mathbf{A}^{-1}\mathbf{B}$ or $\mathbf{LK}$ are diagonalizable. We may obtain the eigenvalues and the corresponding eigenvectors of matrix $\mathbf{LK}$ in (26) by solving equations

$$\det(\mathbf{LK} - \lambda_i \mathbf{I}) = 0$$

and

$$(\mathbf{LK} - \lambda_i \mathbf{I})|e_i\rangle = 0$$

where $|e_i\rangle$ is an arbitrarily-normed i-th voltage eigenvector.

Let $\mathbf{\Omega} = \mathbf{LK}$ and $\mathbf{\Psi} = \mathbf{KL}$; then $\mathbf{\Omega}^t = \mathbf{\Psi}$, due to the symmetry of $\mathbf{K}$ and $\mathbf{L}$. Since the determinant of a matrix is equal to the determinant of its transpose, we have

$$\det(\mathbf{KL} - \lambda\mathbf{I})^t = \det(\mathbf{LK} - \lambda\mathbf{I})$$

i.e.,

$$\det(\mathbf{KL} - \lambda\mathbf{I}) = \det(\mathbf{LK} - \lambda\mathbf{I}).$$

As a consequence, matrix $\mathbf{LK}$ and its transpose, $\mathbf{KL}$, share the same eigenvalues of $\lambda_i = \frac{1}{\nu_i^2}, i = 1, 2, \ldots, n$.

The transformation matrix between modal and circuit voltages, $\mathbf{M}_V$ (or $\mathbf{S}_V$ in [6]), consists of the voltage eigenvectors $\{|e_i\rangle\}$ as its i-th columns. Thus, the unknown vector $|V\rangle$ and matrix $\mathbf{LK}$ can be represented in the $\{|e_i\rangle\}$ basis by

$$\mathbf{M}_V^{-1}|V\rangle = |V'\rangle$$

and

$$\mathbf{M}_V^{-1}\mathbf{LK}\mathbf{M}_V = \mathbf{\Lambda} = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}. \tag{27}$$

Applying these transformations to (5), the decoupled modal vector equation of (5) is obtained as

$$\frac{d^2}{dz^2}|V'\rangle = -\omega^2\mathbf{\Lambda}|V'\rangle. \tag{28}$$

Similarly, from (6), we have

$$\frac{d^2}{dz^2}|I'\rangle = -\omega^2\mathbf{\Lambda}|I'\rangle \tag{29}$$

where $|V'\rangle$ and $|I'\rangle$ are the representations of $|V\rangle$ and $|I\rangle$ in the modal bases, and $\mathbf{\Lambda}$ is the diagonal matrix with the eigenvalues of $\mathbf{LK}$ as its diagonal elements. Transforming these two equations into the original basis, the unknown vectors $|V\rangle$ and $|I\rangle$ will be expressed by (19) and (20), respectively, except that $|e_i'\rangle$ should be replaced by an arbitrarily-normed i-th voltage eigenvector $|e_i\rangle$ of matrix $\mathbf{LK}$, and $|i_i'\rangle$ should be replaced by an arbitrarily-normed i-th current eigenvector $|i_i\rangle$ of matrix $\mathbf{KL}$. In this case, the bi-orthogonality between the voltage and the current eigenvectors is automatically guaranteed, because $\mathbf{LK}$ and $\mathbf{KL}$ are adjoint matrices. Nevertheless, attention must be paid to assigning to each voltage eigenvector an individual scalar and to each current eigenvector an individual scalar as its canonical norm. This is because an eigenvector multiplied by an arbitrary nonzero scalar is also an eigenvector corresponding to the same eigenvalue. In other words, assuming that $\mathbf{N}$ is a nonsingular $n \times n$ diagonal matrix, then

$$(\mathbf{M}_V\mathbf{N})^{-1}\mathbf{LK}(\mathbf{M}_V\mathbf{N}) = \mathbf{\Lambda}. \tag{30}$$

From (27) and (30), however, it is observed that $(\mathbf{M}_V\mathbf{N})^{-1} = \mathbf{M}_V^{-1}$ or $\mathbf{M}_V\mathbf{N} = \mathbf{M}_V$, if and only if $\mathbf{N}$ is an identity matrix. Similar to (27), we may obtain

$$\mathbf{M}_I^{-1}\mathbf{KL}\mathbf{M}_I = \mathbf{\Lambda}.$$

Transposition of the above equation leads to

$$\mathbf{M}_I^t\mathbf{LK}(\mathbf{M}_I^t)^{-1} = \mathbf{\Lambda}.$$

This equation shows that matrix $\mathbf{LK}$ is diagonalized by matrix $(\mathbf{M}_I^t)^{-1}$ and its inverse, i.e., $(\mathbf{M}_I^t)^{-1} = \mathbf{M}_V\mathbf{N}$ or $\mathbf{M}_V = (\mathbf{M}_I^t)^{-1}\mathbf{N}^{-1}$. Thus, in general

$$\mathbf{M}_I^t\mathbf{M}_V \neq \mathbf{I}$$

unless $\mathbf{N}$ is an identity matrix. The arbitrariness of $\mathbf{M}_V$ and $\mathbf{M}_I$ is also discussed in [18].

In Section II, we illustrated that the canonical norms of the eigenvectors $\{|e_i^c\rangle\}$ of matrix $\mathbf{LK}$ assure the correct construction of $\mathbf{Z}_c$. In addition to our method, there are at least three other schemes to set the norms of these eigenvectors:

1) Find *and normalize* the eigenvectors $\{|e_i\rangle\}$ and $\{|i_i\rangle\}$ of matrices $\mathbf{LK}$ and $\mathbf{KL}$.
2) Find *and normalize* the eigenvectors $\{|e_i\rangle\}$, then compute eigenvectors $\{|i_i\rangle\}$ by the bi-orthonormal relation (25).
3) Find *and normalize* the eigenvectors, $\{|e_i\rangle\}$, or leave the norms of $\{|e_i\rangle\}$ arbitrary, then determine the current eigenvectors by means of (1), and compute the characteristic impedance by

$$\mathbf{Z}_c = \mathbf{M}_V\mathbf{\Lambda}^{-1}\mathbf{M}_V^{-1}\mathbf{L}$$

where $\mathbf{M}_I^{-1}$ in (24) has been replaced by $\mathbf{\Lambda}^{-1}\mathbf{M}_V^{-1}\mathbf{L}$, which is obtained from (1).

In Scheme 1, the bi-orthonormality requirement cannot be satisfied unless the $\{|e_i\rangle\}$ themselves form an orthonormal basis, and so do the $\{|i_i\rangle\}$ for $i = 1, 2, \ldots, n$. Furthermore, the canonical norms of $\{|e_i\rangle\}$ or $\{|i_i\rangle\}$ are rarely equal to one. Thus, the constraint between the voltage and current vectors $|V\rangle$ and $|I\rangle$ is rarely satisfied. Therefore, this scheme is rarely valid. It may be observed that (19) and (20) are the general solutions to (5) and (1), respectively. After multiplying V_i, the i-th component of $|V\rangle$, by a nonzero scalar for each i, $|V\rangle$ will still be the general solution to the same equation. However, thereafter, $|I\rangle$ will not remain the solution to (1), because $|V\rangle$ and $|I\rangle$ are constrained through (1) or (2). If either $|V\rangle$ or $|I\rangle$ is fixed, the other must be computed through the corresponding coupling equation.

Scheme 2 guarantees the bi-orthonormality between the basis vectors $\{|e_i\rangle\}$ and $\{|i_i\rangle\}$, but violates the constraint of the general solution to the telegrapher's equations, in general. Examples of erroneous results from improper usage of these two approaches will be provided in the next section.

It may be observed from the impedance expression in Scheme 3 that the arbitrary factors in the norms of the basis vectors $\{|e_i\rangle\}$ have been canceled due to the involvement of both $\mathbf{M}_V$ and $\mathbf{M}_V^{-1}$ in the expression of $\mathbf{Z}_c$ while the orthogonality between $\{|e_i\rangle\}$ and $\{|i_i\rangle\}$ is still preserved. With this approach, (15) becomes $\mathbf{Q}^t\mathbf{A}\mathbf{Q} = \mathbf{D}$ where $\mathbf{D}$ is a diagonal matrix, rather than an identity matrix $\mathbf{I}$. Unlike Schemes 1 and 2, this method does provide correct answers. The only drawback of this method is the deteriorating accuracy of $\mathbf{Z}_c$ due to the computation of the inverse of $\mathbf{M}_V$ and the cancelation of the arbitrary constants involved in the norms when matrix $\mathbf{M}_V$ is nearly ill-conditioned. With this scheme, (24) and (25) are not valid.

IV. Examples, Remarks, and Cautions

In the preceding sections we have discussed the properties of the dual bases in an n-dimensional Euclidean space, and have emphasized the nonorthogonal property of the $\{|e_i\rangle\}$ or $\{|i_i\rangle\}$ and the bi-orthonormal property between the $\{|e_i\rangle\}$ and $\{|i_i\rangle\}$. To provide physical and geometric insight into these properties and to illustrate the features and limitations of linear transformations, we present here a number of examples.

Example 1: Recently, Amari [19] demonstrated an internally inconsistent result where any two symmetric coupled transmission lines could have a diagonal characteristic impedance matrix, and, regardless of their physical and geometric properties, could have the same characteristic impedance matrix, when both $\{|e_i\rangle\}$ and $\{|i_i\rangle\}$ are normalized (Scheme 1). A similar question was posed by Sun [20] and replied to by Marx [21].

In Amari's article, the normalized $\mathbf{M}_V$ and $\mathbf{M}_I$ matrices were as follows

$$\mathbf{M}_I = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

and

$$\mathbf{M}_V = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

The characteristic impedance matrix, $\mathbf{Z}_c$, is obtained as

$$\mathbf{Z}_c = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We see that $\mathbf{Z}_c$ is an identity matrix because both $\mathbf{M}_V$ and $\mathbf{M}_I$ are unitary, i.e., $\mathbf{M}_V^t = \mathbf{M}_V^{-1}$ and $\mathbf{M}_I^t = \mathbf{M}_I^{-1}$. Using (24), we always obtain $\mathbf{Z}_c = \mathbf{I}$.

Taking the symmetric two conductor case as illustrated in [19], and computing M_v with our new algorithm, we obtain the following results

$$\mathbf{M}_v = \begin{pmatrix} -6.32 & 7.01 \\ 6.32 & 7.01 \end{pmatrix}$$

and, using (24)

$$\mathbf{Z}_c = \begin{pmatrix} 88.99 & 9.17 \\ 9.17 & 88.99 \end{pmatrix}.$$

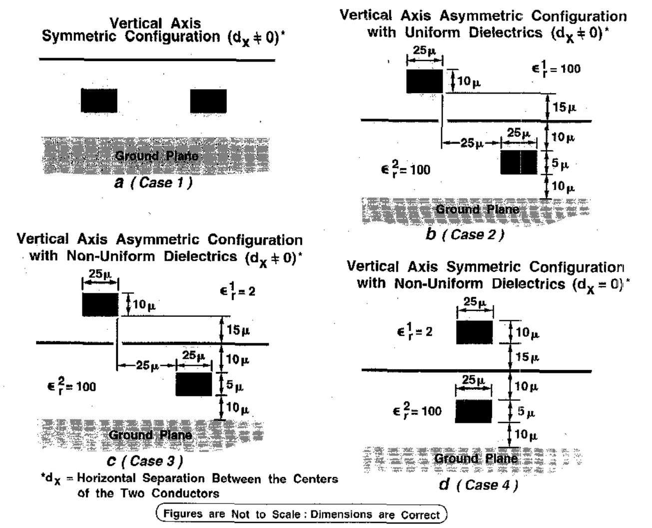

Fig. 1. Two line composite microstrip structures from which the corresponding $\mathbf{L}$ and $\mathbf{K}$ matrices are generated.

It is clear in this example that the two column vectors of $\mathbf{M}_v$ are perpendicular to each other; that is, $\mathbf{M}_v$ is an orthogonal matrix. However, each column vector of $\mathbf{M}_v$ is scaled by $\langle e_i|\mathbf{L}^{-1}|e_i\rangle = \lambda_i$, rather than by $\langle e_i|e_i\rangle = 1$ for $i = 1, 2$, where λ_i is the i-th eigenvalue of matrix $\mathbf{LK}$. This example resolves the heretofore unresolved question raised in Amari's paper.

Example 2: To provide insight into the eigenvector scaling and matrix simultaneous diagonalization, we shall extract the mathematical features of the $\mathbf{LK}$ and $\mathbf{KL}$ matrices of the composite microstrip structures from their corresponding vector-space structures. Fig. 1(a)–1(d) depict a variety of transmission line cross sectional structures of differing physical configurations, and with different material properties. To simplify the problem, the X axis is chosen to be positioned along the juncture between the ground plane and the first dielectric layer. We then see that in the Case 1 structure, the two conductors are symmetric about the Y axis and have the same spacing above the ground plane. In Case 2, the substructures (i.e., the elements of the dielectric sandwich) are identical, but a location for the Y axis cannot be found which makes the structure Y axis symmetric, and the conductors have different spacings above the X axis. Case 3 has the same geometric configuration as that of Case 2, but there is now a lack of homogeneity in the dielectrics as well. Case 4 exhibits Y axis symmetry again, but the two conductors have different heights above the ground plane.

The geometry and dielectric properties of a symmetric embedded microstrip transmission line structure of Fig. 4 in [7], and three asymmetric transmission line structures are depicted in Fig. 1(a)-(d), respectively. Their corresponding inductance and capacitance matrices, $\mathbf{L}$ and $\mathbf{K}$, are for case 1

$$\mathbf{L} = \begin{pmatrix} 494.6 & 63.3 \\ 63.3 & 494.6 \end{pmatrix} \text{ nH/m} \tag{31}$$

$$\mathbf{K} = \begin{pmatrix} 62.8 & -4.9 \\ -4.9 & 62.8 \end{pmatrix} \text{ pF/m.} \tag{32}$$

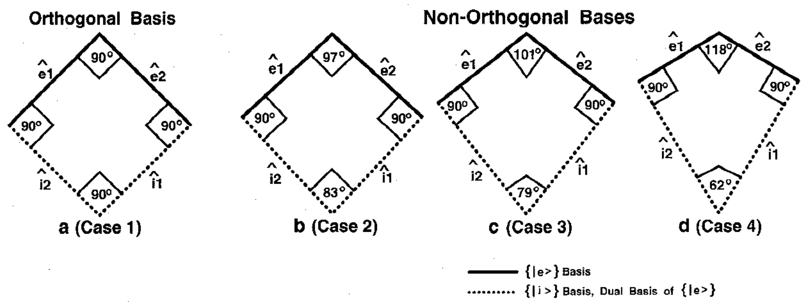

Fig. 2. Geometric representation of the eigen-bases of **LK** and **KL** matrices generated from two-line composite microstrip structures.

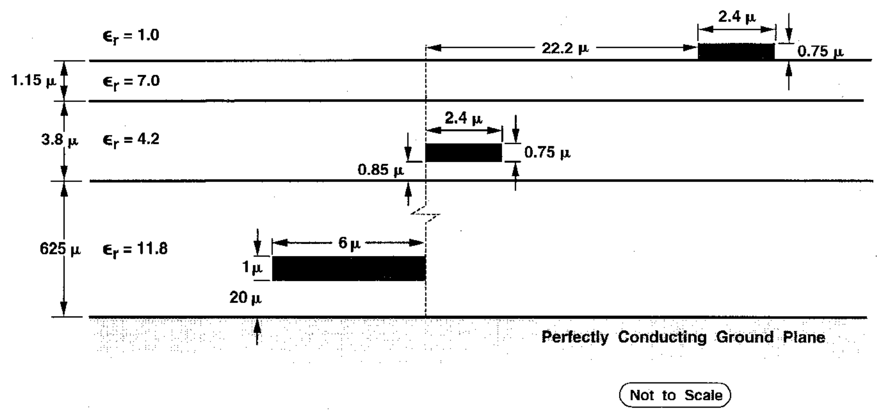

Fig. 3. Composite microstrip structure employed in a numerical demonstration that the eigenvalues of the corresponding **L** and **K** matrices are different from the diagonal elements of the product of the diagonalized **L** matrix and the diagonalized **K** matrix.

For Case 2

$$\mathbf{L} = \begin{pmatrix} 383.04 & 45.31 \\ 45.31 & 232.18 \end{pmatrix} \text{nH/m} \tag{33}$$

$$\mathbf{K} = \begin{pmatrix} 2420.17 & -551.74 \\ -551.32 & 4758.30 \end{pmatrix} \text{pF/m} \tag{34}$$

for Case 3

$$\mathbf{L} = \begin{pmatrix} 383.04 & 45.31 \\ 45.31 & 232.18 \end{pmatrix} \text{nH/m} \tag{35}$$

$$\mathbf{K} = \begin{pmatrix} 143.01 & -22.84 \\ -22.61 & 4239.69 \end{pmatrix} \text{pF/m} \tag{36}$$

and for Case 4

$$\mathbf{L} = \begin{pmatrix} 368.10 & 123.42 \\ 123.38 & 229.87 \end{pmatrix} \text{nH/m} \tag{37}$$

$$\mathbf{K} = \begin{pmatrix} 146.18 & -104.03 \\ -103.84 & 4311.32 \end{pmatrix} \text{pF/m.} \tag{38}$$

The column vectors of $\mathbf{M}_V$ or $\mathbf{M}_I$, $\{|e_i\rangle\}$ or $\{|i_i\rangle\}$, which are the eigenvectors of the **LK** or **KL** matrices, span a two-dimensional Euclidean space. The geometric structures of the $\{|e_i\rangle\}$ and $\{|i_i\rangle\}$ bases in the Euclidean spaces for the four cases are plotted in Fig. 2(a)–(d), respectively, where the spatial angles between every two vectors of the two sets of basis components are computed by the standard directional cosine formula.

In Fig. 2(a), due to the symmetry of the transmission lines, $\{|e_i\rangle\}$ itself forms an orthogonal set, as does $\{|i_i\rangle\}$, i.e., all angles between every pair of these basis vectors equal 90°. This result verifies the bi-orthonormality between the bases $\{|e_i\rangle\}$ and $\{|i_i\rangle\}$ in the original Euclidean space and its dual space, and demonstrates that these two orthogonal coordinate systems are equivalent. In Fig. 2(b), **LK** and **KL** are not symmetric, and the angle between the basis components $|i_1\rangle$ and $|i_2\rangle$ is 83°, while the angle between $|e_1\rangle$ and $|e_2\rangle$ is 97°. Note that the vector $|e_i\rangle$ is no longer parallel to the vector $|i_i\rangle$ for $i = 1, 2$ (the angle between them is not 0°, but 7°). This nonzero angle induces the source of the errors in the conventional normalization schemes, as will be discussed in detail in the final example.

Fig. 2(c) depicts the angle between $|i_1\rangle$ and $|i_2\rangle$ to be 79°, and the angle between $|e_1\rangle$ and $|e_2\rangle$ to be 101°. The angle between the basis components $|i_i\rangle$ and $|e_i\rangle$ for $i = 1, 2$ is 11°. In Fig. 2(d), we see a remarkable increase of the off-orthogonal angles between the "self" bases. The angle between $|i_1\rangle$ and $|i_2\rangle$ is 62°, and the angle between $|e_1\rangle$ and $|e_2\rangle$ is 118°. The angle between the "mutual" bases $|i_i\rangle$ and $|e_i\rangle$ for $i = 1, 2$ is 28°. Nevertheless, the angles between the basis vectors $\{|e_i\rangle\}$ and $\{|i_j\rangle\}$, for $i \neq j$, are always equal to 90°.

For the Case 1 structure, the eigenvectors of the **LK** matrix will not only be linearly independent, but will also be orthogonal to each other. In the asymmetric cases, where the heights above the ground plane for the two conductors are different, decreasing the horizontal separation of the two conductors results in an increasing departure from orthogonality of the spanning set of eigenvectors, *even though* the basis vectors generated from these matrices will continue to be linearly independent. This situation may be understood fairly straightforwardly for the case of a two conductor transmission line. When structures containing three or more conductors are examined, it is virtually impossible that the resulting **KL** or **LK** matrices can generate a spanning set of eigenvectors which are orthogonal, though, as before, they continue to be linearly independent. Thus, in general, the concept of orthogonal basis sets does not apply, and none can be found for those complex structures.

Example 3: Here we shall show a case in which the simultaneous diagonalization scheme of the matrices **L**, **K**, **LK**, and **KL** works for a two symmetric line configuration, but does not work in general.

In Case 1, the diagonal elements of matrix **K** are identical, as are those of matrix **L**. With this property, the 2×2 matrices **L** and **K** as well as the matrices $\mathbf{L}^{-1}$ and **K** commute, which guarantees also that the product of either **LK** or **KL** is a real symmetric matrix. As a result, the eigenvectors of **L** are also eigenvectors of **K**. The necessary and sufficient condition that two real symmetric matrices commute is that they have a common complete set of orthonormal eigenvectors [14, 22]. For this special case, the eigenbasis of $\mathbf{K}^{-1}$ coincides with that of **L**, as well as with that of **LK**, and the transformation matrix $\mathbf{M}_v$ becomes an orthogonal matrix. Furthermore, the matrices **L** and $\mathbf{L}^{-1}$ or **K** and $\mathbf{K}^{-1}$ have a common complete set of orthonormal eigenvectors, because a matrix and its inverse always commute with each other. Thus, the two bases, $\{|e_\imath\rangle\}$ and $\{|i_i\rangle\}$, become equivalent, i.e., the orthogonal bases are self dual. Now it is readily apparent that the six matrices **L**, $\mathbf{L}^{-1}$, **K**, $\mathbf{K}^{-1}$, **LK**, and **KL** can be diagonalized in their common eigenbasis, $\{|e_\imath\rangle\}$ or $\{|i_\imath\rangle\}$. Applying the congruence transformation to the **L** and **K** matrices, we obtain

$$\mathbf{L}_d = \mathbf{M}_I^t\mathbf{L}\mathbf{M}_I = \begin{pmatrix} 431.30 & 0.00 \\ 0.00 & 557.90 \end{pmatrix} \text{nH/m}$$

$$\mathbf{K}_d = \mathbf{M}_V^t\mathbf{K}\mathbf{M}_V = \begin{pmatrix} 67.70 & 0.00 \\ 0.00 & 57.90 \end{pmatrix} \text{pF/m}$$

where $\mathbf{M}_I^t = \mathbf{M}_I^{-1}$, and the diagonal elements of $\mathbf{L}_d$ and $\mathbf{K}_d$ are the eigenvalues of **L** and **K**, respectively.

This validates (7a) and (7b) in [8] which describe the simultaneous diagonalization of the matrices **L** and **K** by the matrix $\mathbf{M}_V$, namely

$$\mathbf{L}_k = \mathbf{M}_V^{-1}\mathbf{L}(\mathbf{M}_V^t)^{-1} = \frac{1}{u_k^2}\mathbf{K}_k^{-1}$$

$$\mathbf{K}_k = \mathbf{M}_V^t\mathbf{K}\mathbf{M}_V$$

where $\mathbf{M}_I = (\mathbf{M}_V^t)^{-1}$. However, if the 2×2 matrices **L** and **K** do not have identical diagonal elements, and thus do not commute, then the bases $\{|e_i\rangle\}$ and $\{|i_\imath\rangle\}$, or equivalently $\mathbf{M}_v$ and $\mathbf{M}_\imath$, are different from each other. If matrices **L** and **K** do not commute, the diagonalization of **L** and **K** by (7a) and (7b) in [8] may lead to erroneous results, as will be demonstrated in the following section.

Applying the congruence transformation to the **L** and **K** matrices, for the first asymmetric case, we obtain

$$\mathbf{L}_d = \mathbf{M}_I^t\mathbf{L}\mathbf{M}_I = \begin{pmatrix} 380.71 & 0.17 \\ 0.18 & 224.47 \end{pmatrix} \text{nH/m}$$

$$\mathbf{K}_d = \mathbf{M}_V^t\mathbf{K}\mathbf{M}_V = \begin{pmatrix} 2331.90 & 2.19 \\ 1.77 & 4730.60 \end{pmatrix} \text{pF/m.}$$

The eigenvalues of **L** and **K** are calculated as 395.599 and 219.624 (nH/m) for **L** and 2296.60 and 4881.87 (pF/m) for **K**, respectively. It may be observed that the diagonal elements of $\mathbf{L}_d$ and $\mathbf{K}_d$ are different from the eigenvalues of **L** and **K**, and the **L** and **K** matrices are not truly diagonalized by the congruence transformation.

For the second asymmetric case, we obtain

$$\mathbf{L}_d = \mathbf{M}_I^t\mathbf{L}\mathbf{M}_I = \begin{pmatrix} 360.19 & 0.02 \\ 0.02 & 232.30 \end{pmatrix} \text{nH/m}$$

$$\mathbf{K}_d = \mathbf{M}_V^t\mathbf{K}\mathbf{M}_V = \begin{pmatrix} 143.07 & 0.24 \\ 0.01 & 4077.80 \end{pmatrix} \text{pF/m.}$$

The eigenvalues of **L** and **K** are calculated as 395.60 and 219.62 (nH/m) for **L**, and 142.88 and 4239.82 (pF/m) for **K**, respectively. It may be observed that the difference between the diagonal elements of the matrices $\mathbf{L}_d$ and $\mathbf{K}_d$ and the eigenvalues of the matrices **L** and **K** becomes slightly larger than that in the first asymmetric case.

For the third or last asymmetric case, we obtain

$$\mathbf{L}_d = \mathbf{M}_I^t\mathbf{L}\mathbf{M}_I = \begin{pmatrix} 235.96 & 0.03 \\ 0.01 & 228.32 \end{pmatrix} \text{nH/m}$$

$$\mathbf{K}_d = \mathbf{M}_V^t\mathbf{K}\mathbf{M}_V = \begin{pmatrix} 145.04 & 0.15 \\ 0.02 & 3315.70 \end{pmatrix} \text{pF/m.}$$

The eigenvalues of **L** and **K** are calculated as 440.42 and 157.55 (nH/m) for **L** and 143.58 and 4313.91 (pF/m) for **K**, respectively. From this case, a large difference between the diagonal elements of the matrices $\mathbf{L}_d$ and $\mathbf{K}_d$ and the eigenvalues of the matrices **L** and **K** is observed. Clearly, all the above $\mathbf{L}_d$ and $\mathbf{K}_d$ matrices are not truly diagonal, though the off-diagonal elements are much smaller than the diagonal elements. This situation occurs because the normalized congruence transformation matrix $\mathbf{M}_I$ ($\mathbf{M}_V$) is not designed to diagonalize **L** (**K**), unless **L** and **K** commute.

The discrepancy between the diagonal elements of $\mathbf{L}_d$ and the eigenvalues of **L**, and between the diagonal elements of $\mathbf{K}_d$ and the eigenvalues of **K**, increases with the off-orthogonal angles between the eigenbases $\{|e_\imath\rangle\}$ or $\{|i_\imath\rangle\}$. From the geometric structure of the bases, it may be observed that for asymmetric lines, the simultaneous diagonalization of matrices **L** and **K** fails. Note that through the use of the new algorithm introduced in Section II, the matrices $\mathbf{L}^{-1}$ and **K** as well as **LK**, and the matrices $\mathbf{K}^{-1}$ and **L** as well as **KL**, can be simultaneously diagonalized by the $\mathbf{M}_v$ and $\mathbf{M}_\imath$ transformations, respectively, with the corresponding canonical norms. However, the diagonal elements of $\mathbf{L}_d$ and $\mathbf{K}_d$ are not the eigenvalues of **L** and **K** (though the diagonal elements of $\mathbf{K}_d$ *are* the eigenvalues of **LK**). It is clear that the diagonal elements of $\mathbf{L}_d\mathbf{K}_d$ will not be the eigenvalues of **LK**, and thus cannot provide the correct values of the modal velocities. If **L** is diagonalized by means of the congruence transformation $\mathbf{M}_I$, and **K** is diagonalized with the congruence transformation of $\mathbf{M}_V$, these two matrices **L** and **K** are transformed into two different coordinate systems. The simultaneous diagonalization scheme is valid only when $\mathbf{M}_V$ and $\mathbf{M}_I$ are orthogonal.

Example 4: To emphasize the conclusion discussed in Example 3, we introduce yet another example for the asymmetric three-line in a multilayer media, as depicted in Fig. 3. The capacitance and the inductance matrices of the composite microstrip structure of Fig. 3 are computed as

$$\mathbf{K} = \begin{pmatrix} 0.6266 & -0.2295 & -0.0130 \\ -0.2295 & 0.4380 & -0.0068 \\ -0.0130 & -0.0068 & 2.1480 \end{pmatrix} \text{pF/cm}$$

and

$$\mathbf{L} = \begin{pmatrix} 14.4600 & 8.0360 & 0.1307 \\ 8.0360 & 14.4700 & 0.1302 \\ 0.1307 & 0.1302 & 6.1020 \end{pmatrix} \text{nH/cm.}$$

The eigenvalues of $\mathbf{LK}$ are computed as 7.3356e-17, 4.3721e-17, and 1.3104e-16 $(\text{sec/m})^2$. After performing the similarity transformations on $\mathbf{L}$ and $\mathbf{K}$, the diagonalized matrices $\mathbf{L}$ and $\mathbf{K}$, namely, $\mathbf{L}_d$ and $\mathbf{K}_d$ turn out to be

$$\mathbf{L}_d = \begin{pmatrix} 2.2503e+01 & 0.0000e+00 & 0.0000e+00 \\ 0.0000e+00 & 6.4289e+00 & 0.0000e+00 \\ 0.0000e+00 & 0.0000e+00 & 6.0999e+00 \end{pmatrix} \text{nH/cm}$$

and

$$\mathbf{K}_d = \begin{pmatrix} 2.8409e-01 & 0.0000e+00 & 0.0000e+00 \\ 0.0000e+00 & 7.8038e-01 & 0.0000e+00 \\ 0.0000e+00 & 0.0000e+00 & 2.1481e+00 \end{pmatrix} \text{pF/cm.}$$

The diagonal matrix of the matrix product L_dK_d is the following

$$\mathbf{L}_d\mathbf{K}_d = \begin{pmatrix} 6.3929e-17 & 0.0000e+00 & 0.0000e+00 \\ 0.0000e+00 & 5.0171e-17 & 0.0000e+00 \\ 0.0000e+00 & 0.0000e+00 & 1.3103e-16 \end{pmatrix} (\text{sec/m}^2).$$

It is clear that the three diagonal elements are not the same as the eigenvalues of $\mathbf{LK}$. These differences occur because the $\mathbf{L}$ and $\mathbf{K}$ matrices were diagonalized in two different coordinate systems. Even though $\mathbf{L}$ and $\mathbf{K}$ are truly diagonalized, the product of the corresponding eigenvalues cannot lead to the correct eigenmode velocities.

Example 5: Numerical results showing the characteristic impedance matrix, $\mathbf{Z}_c$, constructed by the improperly normalized $\{|e_i\rangle\}$ and $\{|i_i\rangle\}$ discussed in Section III, are presented as the final example. As stated earlier, at least three schemes have been used to specify the norms of $\{|e_i\rangle\}$ and $\{|i_i\rangle\}$. However, two of the three yield incorrect results, as will be demonstrated in the following examples:

1) In Section III, we described the normalization for both $\{|e_i\rangle\}$ and $\{|i_i\rangle\}$ as Scheme 1 and discussed a two line symmetric case in Example 2. We shall analyze the third asymmetric line case in Example 2 using this scheme. With the matrices $\mathbf{L}$ and $\mathbf{K}$ given by (37) and (38), the corresponding normalized voltage and current eigenvector matrices are

$$\mathbf{M}_V = \begin{pmatrix} -1.000 & 0.467 \\ -0.006 & 0.884 \end{pmatrix}$$

and

$$\mathbf{M}_I = \begin{pmatrix} -0.884 & -0.006 \\ 0.467 & 1.000 \end{pmatrix}.$$

Thus,

$$\mathbf{M}_I\mathbf{M}_V^t = \begin{pmatrix} 0.881 & -0.001 \\ 0.000 & 0.881 \end{pmatrix}$$

the bi-orthonormal relationship is not satisfied, i.e., $\mathbf{M}_I\mathbf{M}_V^t \neq \mathbf{I}$, because the angles between the basis components $|i_i\rangle$ and $|e_i\rangle$ for $i = 1, 2$ are not equal to $0°$. An inspection of the above matrices reveals that $\langle e_i|i_i\rangle < 1$ in the nonorthogonal $\mathbf{M}_V^t$ and $\mathbf{M}_I$ cases, and the offset between $\mathbf{M}_V^t$ $\mathbf{M}_I$ and $\mathbf{I}$ would become larger when the nonorthogonality of the bases $\{|e_i\rangle\}$ and $\{|i_i\rangle\}$ increases.

The characteristic impedance matrix, $\mathbf{Z}_c$, constructed by (24) turns out to be

$$\mathbf{Z}_c = \begin{pmatrix} 1.383 & 0.476 \\ 0.476 & 0.887 \end{pmatrix}.$$

The characteristic impedance matrix obtained using our scheme to construct $\mathbf{M}_v$ is

$$\mathbf{Z}_c = \begin{pmatrix} 47.829 & 4.178 \\ 4.178 & 7.362 \end{pmatrix}.$$

Comparing these two representations of $\mathbf{Z}_c$, the significant discrepancy between them is apparent.

2) In the second scheme of setting the norms of $\{|e_i\rangle\}$ and $\{|i_i\rangle\}$ described in Section III, (5) was solved and the $\{|e_i\rangle\}$ were normalized, and then the $\{|i_i\rangle\}$ were computed by means of the bi-orthonormal relation (25). The $\mathbf{M}_V$ matrix is the same as that in 1). The corresponding $\mathbf{M}_I$ matrix for the third asymmetric case is

$$\mathbf{M}_I = \begin{pmatrix} -1.003 & -0.008 \\ 0.531 & 1.135 \end{pmatrix}.$$

It may be observed that to satisfy the bi-orthonormality between $\mathbf{M}_V^t$ and $\mathbf{M}_I$, the norms of each current eigenvector must be greater than unity, and the norms must become even greater with further increases in the nonorthogonality of the $\{|e_i\rangle\}$ and $\{|i_i\rangle\}$ bases.

The characteristic impedance matrix constructed by (24) is

$$\mathbf{Z}_c = \begin{pmatrix} 1.218 & 0.419 \\ 0.419 & 0.782 \end{pmatrix}.$$

Similarly, the difference in the $\mathbf{Z}_c$ generated by this approach and by our approach is large. Note that using this approach, the relationship between $\mathbf{M}_V^t$ and $\mathbf{M}_I$ is fixed by (25). However, the necessary constraint established by (1), or equivalently, by $\mathbf{M}_I^{-1} = \mathbf{\Lambda}^{-1}\mathbf{M}_V^{-1}\mathbf{L}$, cannot be satisfied in the general case. Thus, the characteristic impedance matrix constructed by (24) is still incorrect.

V. Conclusion

In this paper, we have developed a novel method to decouple the two-sided matrix differential equations and to specify the canonical norms of the voltage and current eigenvectors for the construction of the characteristic impedance matrix. The core technique of this method was then used to prove the diagonalizability of the nonnormal matrix **LK** (or **KL**); thus, the conventional approach which decouples the one-sided matrix differential equations has now been justified. We also revealed the feature of the nonHermitian matrices **LK** and **KL** that their eigenbases are nonorthogonal. Therefore, the projection theorem and other related frame work developed in the orthonormal basis cannot be directly applied. Based on this work, we have rigorously analyzed the modal decoupling method and have illustrated the limitations in the application of the diagonalization of two or more matrices. A historical problem involved in the normalization of the voltage and current eigenvectors of matrices **LK** and **KL** has been resolved. In addition, we clarified the theory of partial differential equations and provided a procedure of obtaining solutions to the coupled multi-variable differential equations. Finally, we wish to emphasize that under accidental-degeneracy conditions, a complex nonnormal matrix **ZY**, whose counter pair is **LK**, is nondiagonalizable. The solution of this latter problem will be the subject of a future submission.

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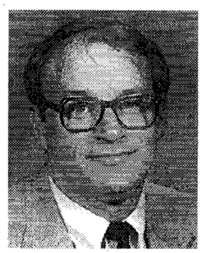

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Dr. Gilbert's research group received the 1994 "ARPA" Director's Award for sustained excellence.